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CONVENIENT PRETOPOLOGIES ON $\mathbb{Z}^2$

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Abstract. We deal with pretopologies on the digital plane $\mathbb{Z}^2$ convenient for studying and processing digital pictures. We introduce a certain natural graph on the vertex set $\mathbb{Z}^2$ whose cycles are eligible for Jordan curves in the digital plane and discuss the pretopologies on $\mathbb{Z}^2$ with respect to which these cycles are Jordan curves.

Keywords: digital plane, Alexandroff pretopology, Khalimsky topology, connectedness graph, Jordan curve.

AMS Subject Classification: Primary 54A05, 54D05, 68U05; Secondary 54B15, 52C45, 05C40.

1. Introduction

To be able to study and process digital images, we need to provide the digital plane $\mathbb{Z}^2$ with a convenient structure. Here, the convenience means that such a structure satisfies analogues of some basic geometric properties of the Euclidean plane $\mathbb{R}^2$. First of all, it is required that an analogue of the Jordan curve theorem is valid (recall that the classical Jordan curve theorem states that every simple closed curve in the Euclidean plane divides the plane into exactly two connected components). In the classical approach (see [9], [10]), graph theoretic tools are used for structuring $\mathbb{Z}^2$, namely the well-known binary relations of 4-adjacency and 8-adjacency. Unfortunately, neither 4-adjacency nor 8-adjacency itself allows an analogue of the Jordan curve theorem, so that a combination of the two adjacencies has to be used. To eliminate this inconvenience, a new, purely topological approach to the problem was proposed in [4] which utilizes the so-called Khalimsky topology for structuring the digital plane. This approach was then developed by many authors, cf. [5] - [8]. In [11], it was shown that it may be advantageous to use closure operators (more general than topologies) for structuring the digital plane. In the present note, we study a special type of such operators on $\mathbb{Z}^2$, namely the pretopologies. We introduce a certain natural graph on the vertex set Z^2 whose cycles are eligible for Jordan curves in $\mathbb{Z}^2$ and we solve the problem of finding pretopologies on $\mathbb{Z}^2$ with respect to which these cycles are Jordan curves. We focus on the minimal of these pretopologies and show that the well-known Khalimsky and Marcus-Wyse topologies and two more convenient pretopologies on $\mathbb{Z}^2$ may be obtained as their quotients.

2. Preliminaries

Throughout the note, topologies are considered to be given by closure operators. Thus, a *topology* on a set X is nothing but a so-called Kuratowski closure operator on X, i.e., a map p: $\exp X \to \exp X$ (where $\exp X$ denotes the power set of X) fulfilling the following four axioms:

(i) $p\emptyset = \emptyset$,
(ii) $A \subseteq pA$ for all $A \subseteq X$,
(iii) $p(A \cup B) = pA \cup pB$ for all $A, B \subseteq X$,
(iv) $ppA = pA$ for all $A \subseteq X$.

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The pair (X, p) is then called a *topological space*. If p satisfies the axioms (i)-(iii) but not necessarily (iv), then it is called a *pretopology* on X (and the pair (X, p) is called *a pretopological space*). Pretopological spaces were studied by E. Cech in [2] and, therefore, they are called Cech closure spaces by some authors.

Basic topological concepts (see e.g. [3]) may naturally be extended from topological spaces to closure ones. Let us recall definitions of some (extended) topological concepts that will particularly be important in this note. If (X, p) is a pretopological space, then a subset $A \subseteq X$ is said to be *closed* if $pA = A$ and it is called *open* if its complement $X - A$ is closed. A pretopology p on a set X is called an *Alexandroff pretopology* if $pA = \bigcup_{x \in A} p\{x\}$ for every $A \subseteq X$ and it is called a T_0-*pretopology* ($T_{\frac{1}{2}}$-*pretopology*) if $x \in p\{y\}$ and $y \in p\{x\}$ imply $x = y$ whenever $x, y \in X$ (if each singleton is closed or open). A pretopological space (X, p) is *connected* if $\emptyset$ and X are the only subsets of X which are both closed and open. A subset $A \subseteq X$ is said to be connected if it is connected as the subspace of (X, p) (i.e., the pretopological space (A, q) where $qB = pB \cap A$ whenever $B \subseteq A$) and it is said to be a *component* if it is a maximal connected subset of X. A map $f : (X, p) \to (Y, q)$ between pretopological spaces (X, p) and (Y, q) is said to be *continuous* if $f(pA) \subseteq q(f(A))$ whenever $A \subseteq X$. If q, r are pretopologies on a set X, we write $q \leq r$ and say that q is *finer* than r (and r is *coarser* than q) if $qA \subseteq rA$ for every $A \subseteq X$. Of course, $\leq$ is a partial order on the set of all pretopologies on X. Given a pretopological space (X, p) and a surjection $e : X \to Y$, a pretopology q on Y is called the *quotient pretopology* of p generated by e if q is the finest pretopology on Y with $e : (X, p) \to (Y, q)$ continuous.

We will use some basic graph-theoretical concepts-we refer for them to [1]. By a *graph* on a set V we always mean an undirected simple graph without loops whose vertex set is V, i.e., a pair (V, E) where $E = \{\{x, y\}; x, y \in A, x \neq y\}$ is the set of edges. For each vertex $x \in V$, we denote by $E(x)$ the set of all vertices adjacent to x, i.e., $E(x) = \{y \in X; \{x, y\} \in E\}$.

The *connectedness graph* of a pretopology p on a set X is the graph on X in which a pair of vertices x, y is adjacent if and only if $x \neq y$ and $\{x, y\}$ is a connected subset of (X, p). If p is an Alexandroff pretopology on a set X, then a subset $A \subseteq X$ is connected in (X, p) if and only if each pair of points of A may be joined by a path in the connectedness graph of p contained in A. Every Alexandroff pretopology is given by its connectedness graph provided that every edge of the graph is incident with a point which is known to be closed or to a point which is known to be open (in which case p is T_0). Indeed, the closure of a closed point consists of just this point, the closure of an open point consists of this point and all points adjacent to it and the closure of a mixed point (i.e., a point that is neither closed nor open) consists of this point and all closed points adjacent to it.

In the sequel, only connected Alexandroff pretopologies on $\mathbb{Z}^2$ will be dealt with. In connectedness graphs of these pretopologies, the closed points will be ringed and the mixed ones boxed (so that the points neither ringed nor boxed will be open-note that no of the points of $\mathbb{Z}^2$ may be both closed and open).

In digital image processing, the well-known 4-adjacency and 8-adjacency graphs are used, i.e., the graphs $(\mathbb{Z}^2, A_4)$ and $(\mathbb{Z}^2, A_8)$ where, for every $(x, y) \in \mathbb{Z}^2$, $A_4(x, y) = \{(x + i, y + j);\ i, j \in \{-1, 0, 1\},\ ij = 0,\ i + j \neq 0\}$ and $A_8(x, y) = A_4(x, y) \cup \{(x + i, y + j);\ i, j \in \{-1, 1\}\}$. Sections of the 4-adjacency graph and the 8-adjacency one are demonstrated in Figure 1.

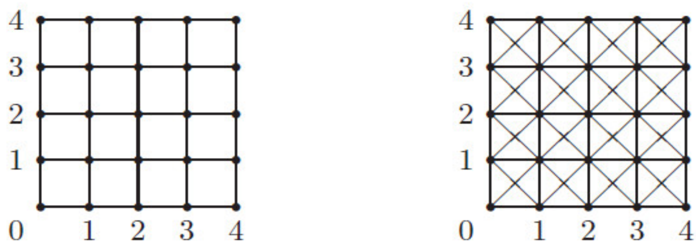

Figure 1. Sections of 4-adjacency graph (on left) and 8-adjacency graph (on right).

For natural reasons related to possible applications of our results in digital image processing, only such pretopologies on $\mathbb{Z}^2$ will be dealt with whose connectedness graphs are subgraphs of the 8-adjacency graph. (It is well known that there are exactly two topologies on $\mathbb{Z}^2$ whose connectedness graphs lie between the 4-adjacency and 8-adjacency graphs. These topologies are the Khalimsky and Marcus-Wyse ones).

The Khalimsky topology on $\mathbb{Z}^2$ is the Alexandroff topology s given as follows:
For any $z = (x, y) \in \mathbb{Z}^2$,

$$s\{z\} = \begin{cases} \{z\} \cup A_8(z) \text{ if } x, y \text{ are even,} \\ \{(x+i, y); i \in \{-1, 0, 1\}\} \text{ if } x \text{ is even and } y \text{ is odd,} \\ \{(x, y+j); j \in \{-1, 0, 1\}\} \text{ if } x \text{ is odd and } y \text{ is even,} \\ \{z\} \text{ otherwise.} \end{cases}$$

The *Khalimsky* topology is connected and T_0; a section of its connectedness graph is shown in Figure 2.

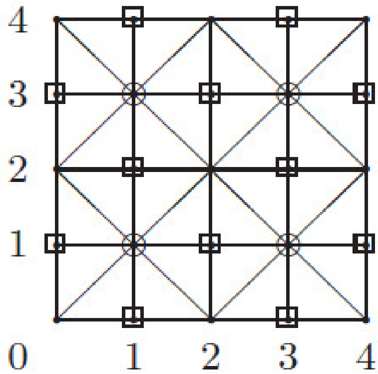

Figure 2. A section of the connectedness graph of the Khalimsky topology.

The *Marcus-Wyse* topology is the Alexandroff topology t on $\mathbb{Z}^2$ given as follows:
For any $z = (x, y) \in \mathbb{Z}^2$,

$$t\{z\} = \begin{cases} \{z\} \cup A_4(z) \text{ if } x + y \text{ is odd,} \\ \{z\} \text{ otherwise.} \end{cases}$$

The Marcus-Wyse topology is connected and $T_{\frac{1}{2}}$. A section of its connectedness graph is shown in Figure 3.

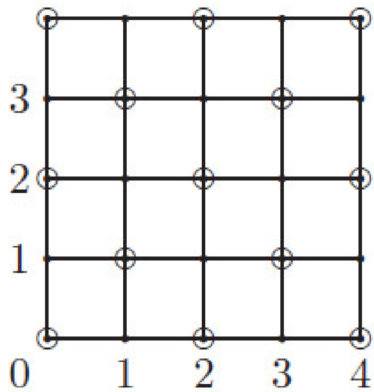

Figure 3. A section of the connectedness graph of the Marcus-Wyse topology.

By a *(digital) simple closed curve* in a pretopological space $(\mathbb{Z}^2, p)$ we mean a nonempty, finite and connected subset $C \subseteq \mathbb{Z}^2$ such that, for each point $x \in C$, there are exactly two points of C adjacent to x in the connectedness graph of p. A simple closed curve C in $(\mathbb{Z}^2, p)$ is said to be a *(digital) Jordan curve* if it separates $(\mathbb{Z}^2, p)$ into precisely two components (i.e., if the subspace $\mathbb{Z}^2 - C$ of $(\mathbb{Z}^2, p)$ consists of precisely two components). Neither 4-adjacency nor 8-adjacency itself allows for an analogue of the Jordan curve theorem so that a combination of the two adjacencies has to be used. This deficiency is eliminated when using the Khalimsky topology on $\mathbb{Z}^2$ because the known Jordan curve theorem for the Khalimsky space proved in [5] says that every simple closed curve with at least four points in the Khalimsky topological space is a Jordan curve. But a Jordan curve in the Khalimsky space cannot turn, at any of its points, to form the acute angle $\frac{\pi}{4}$. It would therefore be useful to replace the Khalimsky topology with certain pretopologies that allow Jordan curves to turn, at some points, to form the acute angle $\frac{\pi}{4}$.

3. Convenient pretopologies on $\mathbb{Z}^2$

Definition 3.1. The *square-diagonal graph* is the graph on $\mathbb{Z}^2$ in which two points $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in \mathbb{Z}^2$ are adjacent if and only if one of the following four conditions is fulfilled:

(1) $|y_1 - y_2| = 1$ and $x_1 = x_2 = 4k$ for some $k \in \mathbb{Z}$,
(2) $|x_1 - x_2| = 1$ and $y_1 = y_2 = 4l$ for some $l \in \mathbb{Z}$;
(3) $x_1 - x_2 = y_1 - y_2 = \pm 1$ and $x_1 - 4k = y_1$ for some $k \in \mathbb{Z}$,
(4) $x_1 - x_2 = y_2 - y_1 = \pm 1$ and $x_1 = 4l - y_1$ for some $l \in \mathbb{Z}$.

A section of the square-diagonal graph is shown in Figure 4.

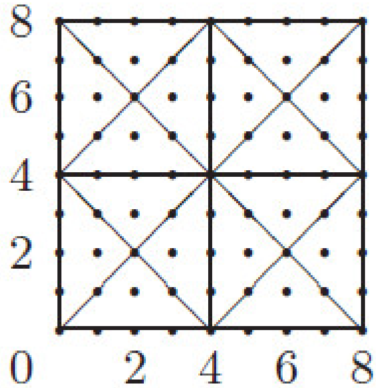

Figure 4. A section of the square-diagonal graph.

When studying digital images, it may be advantageous to equip $\mathbb{Z}^2$ with a pretopology with respect to which all cycles in the square-diagonal graph are Jordan curves.

Definition 3.2. A pretopology p on $\mathbb{Z}^2$ is said to be an *sd-pretopology* if every cycle in the square-diagonal graph is a Jordan curve in $(\mathbb{Z}^2, p)$.

Clearly, neither the Khalimsky topology nor the Marcus-Wyse one is an sd-pretopology (a cycle in the square-diagonal graph is a Jordan curve in the Marcus-Wyse topological space if and only if it does not employ diagonal edges and a cycle in the square-diagonal graph is a Jordan curve in the Khalimsky topological space if and only if it does not turn, at any of its points, at the acute angle $\frac{\pi}{4}$).

Example 3.1. In [12], the Alexandroff $T_{\frac{1}{2}}$-topology w on $\mathbb{Z}^2$ was introduced as follows: For any point $z = (x, y) \in \mathbb{Z}^2$,

$$w\{z\} = \begin{cases} \{z\} \cup A_8(z) \text{ if } x = 4k,\ y = 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - A_4(z)) \text{ if } x = 2 + 4k,\ y = 2 + 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - (\{(x + i, y + 1); i \in \{-1, 0, 1\}\} \cup \{(x, y - 1)\})) \text{ if } x = 2 + 4k, \\ \quad y = 1 + 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - (\{(x + i, y - 1); i \in \{-1, 0, 1\}\} \cup \{(x, y + 1)\})) \text{ if } x = 2 + 4k, \\ \quad y = 3 + 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - (\{(x + 1, y + j); j \in \{-1, 0, 1\}\} \cup \{(x - 1, y)\})) \text{ if } x = 1 + 4k, \\ \quad y = 2 + 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - (\{(x - 1, y + j); j \in \{-1, 0, 1\}\} \cup \{(x + 1, y)\})) \text{ if } x = 3 + 4k, \\ \quad y = 2 + 4l,\ k, l \in \mathbb{Z}, \\ \{(x + i, y); i \in \{-1, 0, 1\}\} \text{ if } x = 2 + 4k,\ y = 4l,\ k, l \in \mathbb{Z}, \\ \{(x, y + j); j \in \{-1, 0, 1\}\} \text{ if } x = 4k,\ y = 2 + 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \text{ otherwise.} \end{cases}$$

A section of the connectedness graphs of w is demonstrated in Figure 5.

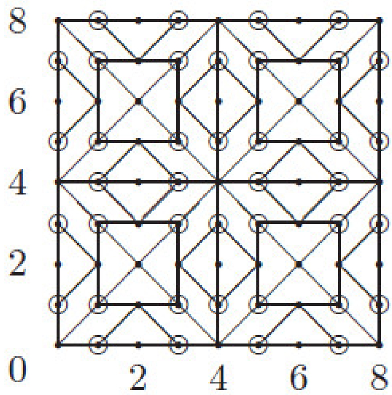

Figure 5. A section of the connectedness graph of w.

It is proved in [12] as the main result that w is an *sd*-pretopology.

Definition 3.3. A graph $(\mathbb{Z}^2, A)$ is said to be *basic* if it has the property that, for every point $z = (x, y) \in \mathbb{Z}^2$,

$$A(z) = \begin{cases} A_8(z) \text{ if } x = 4k, y = 4l,\ k, l \in \mathbb{Z}, \\ (A_8(z) - A_4(z)) \text{ if } x = 2 + 4k,\ y = 2 + 4l,\ k, l \in \mathbb{Z}, \\ \{(x - 1, y), (x + 1, y)\} \text{ if } x = 2 + 4k,\ y = 1 + 2l,\ k, l \in \mathbb{Z}, \\ \{(x, y - 1), (x, y + 1)\} \text{ if } x = 1 + 2k,\ y = 2 + 4l,\ k, l \in \mathbb{Z}, \end{cases}$$

$A(z)$ is a singleton subset of $\{(m, n - 1), (m, n), (m, n + 1)\} \cup V_2(m, n)$ if $z \in \{(m - 1, n), (m + 1, n)\}$ where $m = 4k,\ n = 4l + 2,\ k, l \in \mathbb{Z}$, and
$A(z)$ is a singleton subset of $\{(m - 1, n), (m, n), (m + 1, n)\} \cup H_2(m, n)$ if $z \in \{(m, n - 1), (m, n + 1)\}$ where $m = 4k$, $n = 4l + 2$, $k, l \in \mathbb{Z}$.

A pretopology on $\mathbb{Z}^2$ is called *basic* if its connectedness graph is basic.

Basic graphs are demonstrated in Figure 6. A section of a basic graph is obtained by just choosing, for every vertex demonstrated by a bold dot, exactly one of the three edges denoted by the dashed line segments that are incident with this vertex.

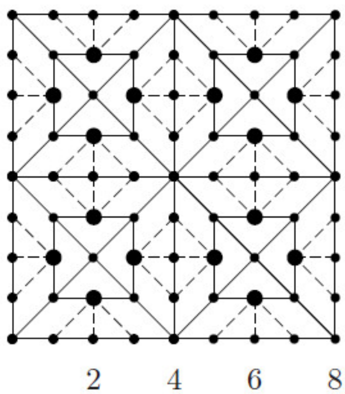

Figure 6. A section of the basic graphs.

Theorem 3.1. *The basic pretopologies are precisely the minimal sd- pretopologies.*

Proof. Let p be a basic pretopology. Clearly, any cycle in the square-diagonal graph is a simple closed curve in $(\mathbb{Z}^2, p)$. Let $z = (x, y) \in \mathbb{Z}^2$ be a point such that $x = 4k + m$ and $y = 4l + n$ for some $k, l, m, n \in \mathbb{Z}$ with $mn = \pm 2$. Then we define the *fundamental triangle $T(z)$* to be the nine-point subset of $\mathbb{Z}^2$ given as follows:

$$T(z) = \begin{cases} \{(r,s) \in \mathbb{Z}^2;\ y-1 \le s \le y+1-|r-x|\} \text{ if } x = 4k+2 \text{ and} \\ \hfill y = 4l+1 \text{ for some } k,l \in \mathbb{Z}, \\ \{(r,s) \in \mathbb{Z}^2;\ y-1+|r-x| \le s \le y+1\} \text{ if } x = 4k+2 \text{ and} \\ \hfill y = 4l-1 \text{ for some } k,l \in \mathbb{Z}, \\ \{(r,s) \in \mathbb{Z}^2;\ x-1 \le r \le x+1-|s-y|\} \text{ if } x = 4k+1 \text{ and} \\ \hfill y = 4l+2 \text{ for some } k,l \in \mathbb{Z}, \\ \{(r,s) \in \mathbb{Z}^2;\ x-1+|s-y| \le r \le x+1\} \text{ if } x = 4k-1 \text{ and} \\ \hfill y = 4l+2 \text{ for some } k,l \in \mathbb{Z}. \end{cases}$$

Graphically, the fundamental triangle $T(z)$ consists of the point z and the eight points lying on the triangle surrounding z - the four types of fundamental triangles are represented in Figure 7.

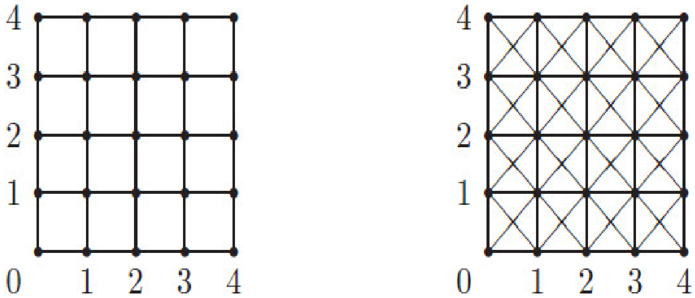

Figure 7. The curve Fundamental triangles.

Given a fundamental triangle, we speak about its sides - it is clear from the above picture what sets are understood to be the sides (note that each side consists of five or three points and that two different fundamental triangles may have at most one common side).

Now, one can easily see that:

(1) Every fundamental triangle is connected (so that the union of two fundamental triangles having a common side is connected) in $(\mathbb{Z}^2, p)$.
(2) If we subtract from a fundamental triangle some of its sides, then the resulting set is still connected in $(\mathbb{Z}^2, p)$.

(3) If S_1, S_2 are fundamental triangles having a common side D, then the set $(S_1 \cup S_2) - M$ is connected in $(\mathbb{Z}^2, p)$ whenever M is the union of some sides of S_1 or S_2 different from D.

(4) Every connected subset of $(\mathbb{Z}^2, p)$ having at most two points is a subset of a fundamental triangle.

We will now show that the following is also true:

(5) For every cycle C in the square-diagonal graph, there are sequences $\mathcal{S}_F, \mathcal{S}_I$ of fundamental triangles, $\mathcal{S}_F$ finite and $\mathcal{S}_I$ infinite, such that, whenever $\mathcal{S} \in \{\mathcal{S}_F, \mathcal{S}_I\}$, the following two conditions are satisfied:

(a) Each member of $\mathcal{S}$, excluding the first one, has a common side with at least one of its predecessors.

(b) C is the union of those sides of fundamental triangles from $\mathcal{S}$ that are not shared by two different fundamental triangles from $\mathcal{S}$.

Put $C_1 = C$ and let S_1^1 be an arbitrary fundamental triangle with $S_1^1 \cap C_1 \neq \emptyset$. For every $k \in \mathbb{Z}$, $1 \leq k$, if $S_1^1, S_2^1, ..., S_k^1$ are defined, let S_{k+1}^1 be a fundamental triangle with the following properties: $S_{k+1}^1 \cap C_1 \neq \emptyset$, S_{k+1}^1 has a side in common with S_k^1 which is not a subset of C_1 and $S_{k+1}^1 \neq S_i^1$ for all i, $1 \leq i \leq k$. Clearly, there will always be a (smallest) number $k \geq 1$ for which no such a fundamental triangle S_{k+1}^1 exists. We denote by k_1 this number so that we have defined a sequence $(S_1^1, S_2^1, ..., S_{k_1}^1)$ of fundamental triangles. Let C_2 be the union of those sides of fundamental triangles from $(S_1^1, S_2^1, ..., S_{k_1}^1)$ that are disjoint from C_1 and are not shared by two different fundamental triangles from $(S_1^1, S_2^1, ..., S_{k_1}^1)$. If $C_2 \neq \emptyset$, we construct a sequence $(S_1^2, S_2^2, ..., S_{k_2}^2)$ of fundamental triangles in an analogous way to $(S_1^1, S_2^1, ..., S_{k_1}^1)$ by taking C_2 instead of C_1 (and obtaining k_2 analogously to k_1). Repeating this construction, we get sequences $(S_1^3, S_2^3, ..., S_{k_3}^3)$, $(S_1^4, S_2^4, ..., S_{k_4}^1)$, etc. We put $\mathcal{S} = (S_1^1, S_2^1, ..., S_{k_1}^1, S_1^2, S_2^2, ..., S_{k_2}^2, S_1^3, S_2^3, ..., S_{k_3}^3, ...)$ if $C_i \neq \emptyset$ for all $i \geq 1$ and $\mathcal{S} = (S_1^1, S_2^1, ..., S_{k_1}^1, S_1^2, S_2^2, ..., S_{k_2}^2, ..., S_1^l, S_2^l, ..., S_{k_l}^l)$ if $C_i \neq \emptyset$ for all i with $1 \leq i \leq l$ and $C_i = \emptyset$ for $i = l + 1$.

Further, let $S_1' = T(z)$ be a fundamental triangle such that $z \notin S$ whenever S is a member of $\mathcal{S}$. Having defined S_1', let $\mathcal{S}' = (S_1', S_2', ...)$ be a sequence of fundamental triangles defined analogously to $\mathcal{S}$ (by taking S_1' in the role of S_1^1). Then one of the sequences $\mathcal{S}$, $\mathcal{S}'$ is finite and the other is infinite. (Indeed, $\mathcal{S}$ is finite or infinite, respectively, if and only if its first member equals such a fundamental triangle $T(z)$ for which $z = (k, l) \in \mathbb{Z}^2$ has the property that (1) k is even, l is odd and the cardinality of the set $\{(x, l) \in \mathcal{Z}^2;\ x > k\} \cap C$ is odd or even, respectively or (2) k is odd, l is even and the cardinality of the set $\{(k, y) \in \mathcal{Z}^2;\ y > l\} \cap C$ is odd or even, respectively. The same is true for $\mathcal{S}'$.) If we put $\{\mathcal{S}_F, \mathcal{S}_I\} = \{\mathcal{S}, \mathcal{S}'\}$ where $\mathcal{S}_F$ is finite and $\mathcal{S}_I$ is infinite, then the conditions (a) and (b) are clearly satisfied.

Given a cycle C in the square-diagonal graph, let S_F and S_I denote the union of all members of $\mathcal{S}_F$ and $\mathcal{S}_I$, respectively. Then $S_F \cup S_I = \mathbb{Z}^2$ and $S_F \cap S_I = C$. Let $\mathcal{S}_F^*$ and $\mathcal{S}_I^*$ be the sequences obtained from $\mathcal{S}_F$ and $\mathcal{S}_I$ by subtracting C from each member of $\mathcal{S}_F$ and $\mathcal{S}_I$, respectively. Let S_F^* and S_I^* denote the union of all members of $\mathcal{S}_F^*$ and $\mathcal{S}_I^*$, respectively. Then S_F^* and S_I^* are connected by (1), (2) and (3) and it is clear that $S_F^* = S_F - C$ and $S_I^* = S_I - C$. So, S_F^* and S_I^* are the two components of $\mathbb{Z}^2 - C$ by (4) ($S_F - C$ is the so-called *inside* component and $S_I - C$ is the so-called *outside* component). We have shown that p is an *sd*-pretopology.

To prove p is a minimal *sd*-pretopology, let q be an *sd*-pretopology such that $q \leq p$. Suppose that there are points $z_1, z_2 \in \mathbb{Z}^2$ such that $z_2 \in p\{z_1\} - q\{z_1\}$. Then the edge $\{z_1, z_2\}$ belongs to the connectedness graph of p but does not belong to the connectedness graph of q. Since the connectedness graph of q is a supergraph of the square-diagonal graph, there is a fundamental triangle $T(z)$ with $z \in \{z_1, z_2\}$. Thus, $\{z_1, z_2\}$ is one of the three edges incident with z and the point $z' \in \{z_1, z_2\} - \{z\}$ lies on a side D of $T(z)$. Let S be the fundamental triangle different from $T(z)$ such that one of the sides of S is D. Then the union C of all sides of $T(z)$ and S different from D is a cycle in the square diagonal graph but it is not a Jordan curve in $(\mathbb{Z}^2, q)$ because the inside part of C, i.e., the set $(T(z) \cup S) - C$, is evidently not connected in the subgraph $\mathbb{Z}^2 - C$ of the connectedness graph of q. Thus, the subgraph $\mathbb{Z}^2 - C$ of the connectedness graph of q has more than two components. This is a contradiction. Therefore, $p = q$ and the minimality of $(\mathbb{Z}^2, p)$ is proved.

Example 3.2. Let r be the Alexandroff T_0-pretopology on $\mathbb{Z}^2$ given as follows:
For any point $z = (x, y) \in \mathbb{Z}^2$,

$$r\{z\} = \begin{cases} \{z\} \cup A_8(z) \text{ if } x = 4k,\ y = 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \cup (A_8(z) - A_4(z)) \text{ if } x = 2 + 4k,\ y = 2 + 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \cup \{(x-1, y), (x+1, y)\} \text{ if } x = 2 + 4k,\ y = 1 + 2l,\ k, l \in \mathbb{Z}, \\ \{z\} \cup \{(x, y-1), (x, y+1)\} \text{ if } x = 1 + 2k, \text{ and } y = 2 + 4l), \\ \{z\} \cup A_4(z) \text{ if either } x = 4k \text{ and } y = 2 + 4l \text{ or } x = 2 + 4k \text{ and} \\ \qquad\qquad y = 4l,\ k, l \in \mathbb{Z}, \\ \{z\} \text{ otherwise.} \end{cases}$$

A section of the connectedness graphs of r is demonstrated in Figure 8.

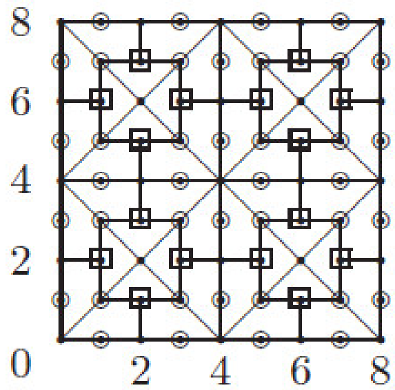

Figure 8. The curve $\alpha(s)$.

It was proved in [14] that r is an sd-pretopology and it is evident that r is even basic.

4. Quotients of the basic pretopologies

We have shown that basic pretopologies possess a rich enough variety of Jordan curves and may therefore be used for structuring the digital plane. We will show that certain convenient pretopologies on $\mathbb{Z}^2$ may be obtained as quotients of the basic pretopologies. First of all, both the Khalimsky and Marcus-Wyse topologies may be obtained in such a way.

The following two theorems may be proved similarly to Theorems 10 and 11 from [13] that state that Khalimsky and Marcus-Wyse topologies are quotients of the topology w (see Example 3.3).

Theorem 4.1. *The Khalimsky topology is the quotient pretopology of any of the basic pretopologies generated by the surjection $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ given as follows:*

$$f(x, y) = \begin{cases} (2k, 2l) \text{ if } (x, y) = (4k, 4l),\ k, l \in \mathbb{Z}, \\ (2k, 2l+1) \text{ if } (x, y) \in A_4(4k, 4l+2),\ k, l \in \mathbb{Z}, \\ (2k+1, 2l) \text{ if } (x, y) \in A_4(4k+2, 4l),\ k, l \in \mathbb{Z}, \\ (2k+1, 2l+1) \text{ if } (x, y) \in \{(4k+2, 4l+2)\} \cup \\ \quad (A_8(4k+2, 4l+2) - A_4(4k+2, 4l+2)),\ k, l \in \mathbb{Z}. \end{cases}$$

The decomposition of the basic pretopological space $(\mathbb{Z}^2, r)$ given by f is demonstrated in Figure 9 by the dashed lines. Every class of the decomposition is mapped by f to its center point expressed by the bold coordinates.

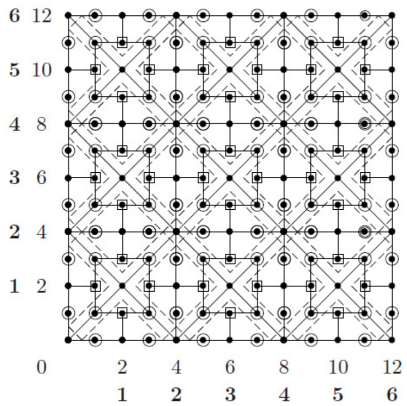

Figure 9. The decomposition of $(\mathbb{Z}^2, r)$ given by f.

Theorem 4.2. *The Marcus-Wyse topology is the quotient pretopology of any of the basic pretopologies generated by the surjection $d : \mathbb{Z}^2 \to \mathbb{Z}^2$ given as follows:*

$$d(x,y) = \begin{cases} (k+l, l-k) \text{ if } (x,y) \in \{(4k,4l)\} \cup A_8(4k,4l), \ k,l \in \mathbb{Z}, \\ (k+l+1, l-k) \text{ if } (x,y) = (4k+2, 4l+2) \text{ for some } k,l \in \mathbb{Z} \\ \qquad \text{with } k+l \text{ odd or } (x,y) \in \{(p,q) \in \mathbb{Z}^2; \ \text{both } x = 4k+2 \\ \qquad \text{and } |y-4l-2| \le 3 \text{ or both } |x-4k-2| \le 3 \text{ and} \\ \qquad y = 4l+2\} \text{ for some } k,l \in \mathbb{Z} \text{ with } k+l \text{ even.} \end{cases}$$

The decomposition of the pretopological space $(\mathbb{Z}^2, r)$ given by d is demonstrated in Figure 10 by the dashed lines. Every class of the decomposition is mapped by d to its center point expressed by the coordinates with respect to the diagonal axes (where the first coordinate relates to the axis with only the non-negative part displayed).

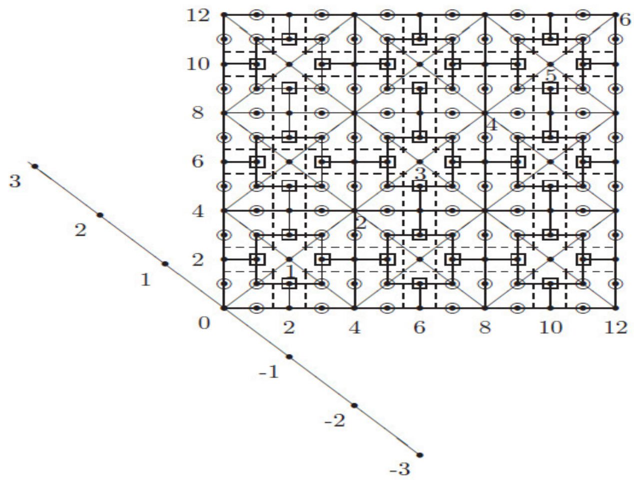

Figure 10. The decomposition of $(\mathbb{Z}^2, w)$ given by g.

We will show now that there are two more convenient Alexandroff pretopologies on $\mathbb{Z}^2$ that may be obtained as quotients of the basic pretopologies.

Let u be the Alexandroff pretopology on $\mathbb{Z}^2$ given as follows:

For any $z = (x,y) \in \mathbb{Z}^2$,

$$u\{z\} = \begin{cases} \{(x+i, y); \ i \in \{-1,0,1\}\} \text{ if } x \text{ is odd and } y \text{ is even}, \\ \{(x, y+j); \ j \in \{-1,0,1\}\} \text{ if } x \text{ is even and } y \text{ is odd}, \\ \{z\} \cup (A_8(z) - A_4(z)) \text{ if } x,y \text{ are odd}, \\ \{z\} \text{ if } x,y \text{ are even.} \end{cases}$$

Evidently, u is a connected and $T_{\frac{1}{2}}$-topology. A portion of its connectedness graph is shown in Figure 11.

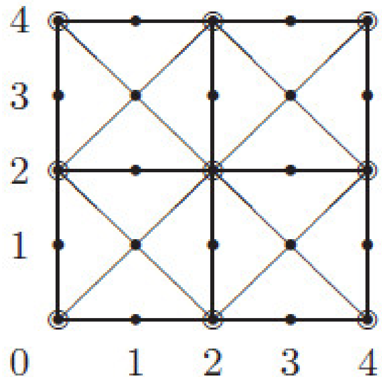

Figure 11. A section of the connectedness graph u.

The following Jordan curve theorem for u is proved in [15] (Proposition 2):

Theorem 4.3. *Let D be a simple closed curve in $(\mathbb{Z}^2, u)$ having more than four points and such that every pair of different points $z_1, z_2 \in D$ with both coordinates even satisfies $A_4(z_1) \cap A_4(z_2) \subseteq D$. Then D is a Jordan curve in $(\mathbb{Z}^2, u)$.*

The proof of the following theorem is analogous to that of Theorem 12 of [13]:

Theorem 4.4. *u is the quotient pretopology of any of the basic pretopologies generated by the surjection $h : \mathbb{Z}^2 \to \mathbb{Z}^2$ given as follows:*

$$h(x,y) = \begin{cases} (2k, 2l) \ \textit{if} \ (x,y) \in \{(4k,4l)\} \cup A_8(4k,4l), \ k,l \in \mathbb{Z}, & \\ (2k, 2l+1) \ \textit{if} \ (x,y) \in \{(4k+i, 4l+2)\}; \ i \in \{-1,0,1\}\}, & k,l \in \mathbb{Z}, \\ (2k+1, 2l) \ \textit{if} \ (x,y) \in \{(4k+2, 4l+j)\}; \ j \in \{-1,0,1\}\}, & k,l \in \mathbb{Z}, \\ (2k+1, 2l+1) \ \textit{if} \ (x,y) = (4k+2, 4l+2), \ k,l \in \mathbb{Z}. & \end{cases}$$

The decomposition of the pretopological space $(\mathbb{Z}^2, r)$ given by h is demonstrated in the Figure 12 by the dashed lines. All points of a class of the decomposition are mapped by h to the center point of the class given by the bold coordinates.

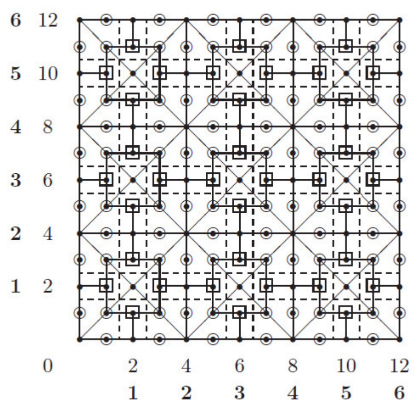

Figure 12. The decomposition of $(\mathbb{Z}^2, r)$ given by h.

Let v be the Alexandroff pretopology on $\mathbb{Z}^2$ given as follows:

For any $z = (x, y) \in \mathbb{Z}^2$,

$$v\{z\} = \begin{cases} \{z\} \cup A_4(z) \text{ if both } x \text{ and } y \text{ are odd or } (x,y) = (4k+2l, 2l+2),\ k,l \in \mathbb{Z}, \\ \{z\} \cup A_8(z) \text{ if } (x,y) = (4k+2l, 2l),\ k,l \in \mathbb{Z}, \\ \{z\} \text{ otherwise.} \end{cases}$$

Evidently, v is connected and T_0. A section of its connectedness graph are shown in Figure 13.

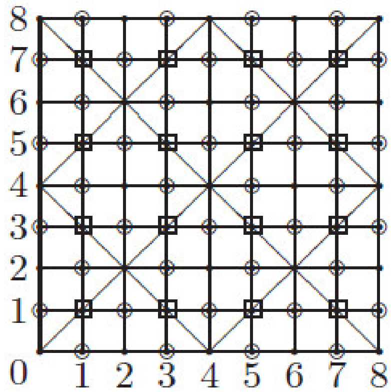

Figure 13. A section of the connectedness graph v.

The following Jordan curve theorem for v immediately follows from [13] (Theorem 8):

Theorem 4.5. *Let D be a simple closed curve in $(\mathbb{Z}^2, v)$ such that, for every point $z \in D$ with both coordinates odd, $A_4(z) \cap D = \emptyset$. Then D is a Jordan curve in $(\mathbb{Z}^2, v)$.*

The proof of the following theorem is analogous to that of Theorem 14 from [13]:

Theorem 4.6. *v is the quotient pretopology of any of the basic pretopologies generated by the surjection $d : \mathbb{Z}^2 \to \mathbb{Z}^2$ given as follows:*

$$d(x,y) = \begin{cases} (2k+2l+1, 2l-2k+1) \text{ if } (x,y) \in A_4(4k, 4l+2),\ k,l \in \mathbb{Z}, \\ (2k+2l+1, 2l-2k-1) \text{ if } (x,y) \in A_4(4k+2, 4l),\ k,l \in \mathbb{Z}, \\ (\frac{x+y}{2}, \frac{y-x}{2}) \text{ if } x, y \text{ are odd or } (x,y) = (4k+2l, 2l),\ k,l \in \mathbb{Z}. \end{cases}$$

The decomposition of the pretopological space $(\mathbb{Z}^2, r)$ given by d is demonstrated in Figure 14 by the dashed lines. Every class of the decomposition is mapped by d to its center point expressed by the coordinates with respect to the diagonal axes (where the first coordinate relates to the axis with only the non-negative part displayed).

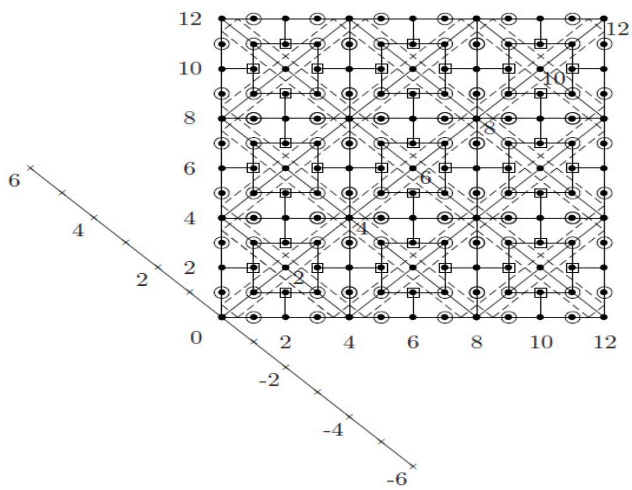

Figure 14. The decomposition of $(\mathbb{Z}^2, r)$ given by d.

5. Acknowledgement

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